Lec 7: 09/09/2009

Expectation Value and Uncertainty:

Consider a particle with state vector $|\psi\rangle$. The expectation value (or mean value) of an observable that is represented by the Hermitian operator Ω is:

$$\langle\Omega\rangle = \sum_{\omega} P(\omega)\,\omega \qquad \text{where} \qquad \Omega|\omega\rangle = \omega|\omega\rangle$$

$$P(\omega) = |\langle\omega|\psi\rangle|^2 \Rightarrow \langle\Omega\rangle = \sum_{\omega} \langle\psi|\omega\rangle\langle\omega|\psi\rangle\,\omega$$

$$= \sum_{\omega} \langle\psi|\omega\rangle\langle\omega|\Omega|\psi\rangle = \langle\psi|\underbrace{\left(\sum_{\omega}|\omega\rangle\langle\omega|\right)}_{I}\Omega|\psi\rangle =$$

$$\langle\psi|\Omega|\psi\rangle$$

Measuring the observable with an ensemble of N particles, prepared in the same state $|\psi\rangle$, or measuring it N times with a single particle in the same state $|\psi\rangle$, yields $\langle\Omega\rangle$ (for large N).

Uncertainty, or standard deviation, in an observable

that is represented by the Hermitian operator

is defined as:

$$\Delta\Omega = \left[\langle\psi|(\Omega - \langle\Omega\rangle)^2|\psi\rangle\right]^{1/2} = \left[\langle\psi|\Omega^2|\psi\rangle - \langle\psi|\Omega|\psi\rangle^2\right]^{1/2}$$

If $|\psi\rangle$ is an eigenvector of Ω, then:

$$\Omega|\psi\rangle = \omega|\psi\rangle \Rightarrow \langle\psi|\Omega^2|\psi\rangle = \omega^2\langle\psi|\psi\rangle = \omega^2$$

$$\langle\psi|\Omega|\psi\rangle^2 = \omega^2\langle\psi|\psi\rangle = \omega^2$$

Thus;

$$\Delta\Omega = 0$$

Density Matrix:

For a particle in state (pure state) $|\psi\rangle$, the density

matrix ρ is defined as the operator:

$$\rho = |\psi\rangle\langle\psi|$$

It has the following properties:

$\rho^\dagger = \rho \qquad Tr\,\rho = 1 \qquad \rho^2 = \rho$

Now:

$Tr(\Omega\rho) = \sum_i \langle i|\Omega|\Psi\rangle\langle\Psi|i\rangle$

Where $|i\rangle$ form an orthonormal basis. If we choose the basis consisting of eigenvectors of Ω:

$Tr(\Omega\rho) = \sum_\omega \langle\omega|\Omega|\Psi\rangle\langle\Psi|\omega\rangle = \sum_\omega \omega\,|\langle\omega|\Psi\rangle|^2$

$= \langle\Omega\rangle$

Density matrix contains all of the statistical information about the state $|\Psi\rangle$.

<u>The Schrodinger Equation and Time Evolution:</u>

The Schrodinger equation is:

$i\hbar \frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$

If H is time-independent, then we have:

$|\Psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\Psi(0)\rangle$

where $|\Psi_{(0)}\rangle$ is the state vector at the intial time $t=0$. Therefore if we know $|\Psi_{(0)}\rangle$, we will know $|\Psi_{(t)}\rangle$ for any t.

The operator $e^{-\frac{iHt}{\hbar}}$ is called the propagator and denoted by U.

Time evolution can be most conveniently described in the basis consisting of eigenvector of the Hamiltonian H:

$H|E\rangle = E|E\rangle$

$|\Psi_{(0)}\rangle = \sum_E a_{(0)} |E\rangle$

$|\Psi_{(t)}\rangle = e^{-\frac{iHt}{\hbar}} |\Psi_{(0)}\rangle = e^{-\frac{iHt}{\hbar}} \sum_E a_{E(0)} |E\rangle =$

$\sum_E a_{E(0)} e^{-\frac{iHt}{\hbar}} |E\rangle = \sum_E a_{E(0)} e^{-\frac{iEt}{\hbar}} |E\rangle$

Once we know $a_{E(0)}$, then we will know $|\Psi_{(t)}\rangle$.

This is why finding the eigenvalues and

eigenvectors of the Hamiltonian are so useful.

The probability to find value E_1 for the energy is:

$$P_{(E_1)} = |\langle E_1 | \Psi \rangle|^2 = |\langle E_1 | \sum_E a_{E}(0) e^{-\frac{iEt}{\hbar}} |E\rangle|^2$$

$$= |a_{E_1}(0)|^2$$

This implies that $P_{(E_1)}$ is time-independent.

If the initial state is one of the eigenvectors of the Hamiltonian:

$$|\Psi(0)\rangle = |E_1\rangle$$

we have:

$$|\Psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |E_1\rangle = e^{-\frac{iE_1 t}{\hbar}} |E_1\rangle$$

But this is just a pure phase. Therefore the system will always remain in the same state.